End-Semester Exam ISI Bangalore November 2019

Algebra I 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(c) $\mathbb{Z}_n$ = the set of integers modulo n, $(\mathbb{Z}_n)^\times$ = the set of those integers in $\mathbb{Z}_n$ that are coprime to n.

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1. [12 points] Let a, b, c be elements of a group.
   (i) Suppose $abc = 1$. Prove or disprove using an example:
   1) $bca = 1$ 2) $bac = 1$.
   (ii) If a has order m, and $a^n b = ba^n$ for some n coprime to m then prove that $ab = ba$.

2. [10 points] Find 3 abelian groups G_1, G_2, G_3 of order 27 such that G_i is not isomorphic to G_j for $i \neq j$.

3. [16 points] Let G be a group and H, N subgroups with N normal in G.
   (i) Verify that $NH = \{nh | n \in N, h \in H\}$ is a subgroup of G.
   (ii) Verify that $H \cap N$ is a normal subgroup of H.
   (iii) Prove that there is a natural isomorphism $H/(H \cap N) \xrightarrow{\sim} NH/N$.

4. [15 points] Let G be the dihedral group of distance-preserving symmetries of a regular hexagon with vertices, say $v_1, v_2, v_3, v_4, v_5, v_6$.
   (i) Find a generating set for G and list the elements of G as products of multiplicative powers of the elements of these generators.
   (ii) Write each element $g \in G$ from (i) as a permutation in S_6 using the way g permutes the vertices v_i.
   (iii) Let d_{ij} denote a diagonal of the hexagon where d_{ij} is the segment joining v_i to v_j with v_i and v_j non-adjacent. For the obvious natural action of G on the set of all diagonals, find the stabilizer of d_{13} and of d_{14}. (Use the permutation notation of (ii) to list the elements of the stabilizers.)

5. [12 points] Prove that any group of order 35 is cyclic.

6. [20 points] Consider the elements $a = (1\,2\,3)$ and $b = (1\,2\,3\,4\,5)$ of $G = S_5$.

(i) Calculate the sizes of the conjugacy classes $|C_G(a)|$ and $|C_G(b)|$. (You don't have to list the elements of the classes.)

(ii) Find the centralizers $Z_G(a), Z_G(b)$ by writing down their elements.

(iii) Let $H = A_5$ denote the alternating group in S_5. Calculate the centralizers $Z_H(a), Z_H(b)$.

(iv) Calculate the sizes of the conjugacy classes $|C_H(a)|$ and $|C_H(b)|$.

7. [15 points] Let G be a group of order 640. Prove that there is a normal subgroup N in G such that $16 \leq |N| \leq 128$.